TABLE 1. COMPARISON OF EQUATION (2) WITH EQUATION (1) AND DATA

Velocity, N_{Re}	Analytical expressions f(Ergun), Equation (1)	f(this work), Equation (2)	f(data plots) Ergun (1, 2)	Tallmadge (3)
10^0	152	154	~150	—
10^1	16.8	17.9	16 to 18	—
10^2	3.25	3.44	3.0 to 3.8	—
10^3	1.90	1.48	1.5 to 1.9	—
$3 \cdot 10^3$	1.80	1.06	—	1.1 to 1.3
10^4	1.77	0.92	—	0.8 to 1.0
$6 \cdot 10^4$	1.76	0.67	—	0.6 to 0.7
10^5	1.75	0.62	—	—

of N_{Re} by extension of the Blake-Kozeny equation to N_{Re} of 10^5. Noting that the data have been described by Tallmadge (3) as the smooth function $f = 5.28\, N_{Re}^{-0.19}$, we obtain the following desired result:

$$f = \frac{150}{N_{Re}} + \frac{a}{N_{Re}{}^{b}} = \frac{150}{N_{Re}} + \frac{4.2}{N_{Re}{}^{1/6}} \tag{2}$$

Here, Equation (2) is valid for $10^{-1} < N_{Re} < 10^5$ (see Table 1).

Other parameters might be used in Equation (2), but it was found that the deviations between data and the values predicted by Equation (2) are larger with $b = 1/5$. With $b = 1/7$, agreement is intermediate to that for 1/5 and 1/6. Thus b was chosen as 1/6. The parameter a was then determined by averaging local a values over the N_{Re} range of the Wentz and Thodos data (2,500 to 65,000).

The high N_{Re} data of Wentz and Thodos were taken over a wide range of porosity (35 to 88%) by using extended beds as well as closely packed beds of spheres. Thus, Equation (2) has been tested for wide ranges of both porosity and Reynolds number. Equation (2) is also an improvement because it includes the small but noticeable effect of N_{Re} in the turbulent region; no such behavior was included in Equation (1).

LITERATURE CITED

1. Bird, R. B., L. N. Lightfoot, and W. E. Stewart, "Transport Phenomena," Figure 6.4-1, Wiley, New York (1960).
2. Ergun, Sabri, *Chem. Eng. Progr.*, **48**, 93 (1952). See also Bird, R. B., et al. (*1*).
3. Tallmadge, J. A., *AIChE J.*, **13**, 599 (1967).
4. Wentz, C. A., and George Thodos, *ibid.*, **9**, 81 (1963).

Single- and Two-Phase Film Flow on Near Horizontal Planes

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The quest for more efficient three-phase heat exchangers for water desalination plants has led to the study of two-phase, liquid-liquid, films flowing concurrently down an inclined plane (*1, 2*). It was, however, experimentally found (*3*) that two-phase flow on a negatively inclined plane with a very thin layer of the lighter upper liquid phase is advantageous, since unlike the flow down a positively inclined plane, the upper film remains unbroken, at identical flow rates.

An analytical solution for two-dimensional flow of films of decreasing thickness on a horizontal plane was given by Nedderman (*4*), including a numerical solution for the case of flow down a plane with a slight positive inclination. The solution is based on the experimentally established (*5*) existence of a parabolic velocity profile in such films.

Analytical integral type of solutions are presented here for single-phase laminar flow of films of varying thickness on near horizontal planes with positive and negative inclinations (Figure 1). These solutions are then extended for the practically interesting case of two-phase, liquid-liquid concurrent flow on a negatively inclined plane with a very thin upper film.

SINGLE-PHASE FLOW

It is assumed that the flow is stable and fully developed and that semiparabolic velocity profile prevails throughout. It is further assumed that the x directed motion is dominating the flow phenomenon, and that the large interfacial radius of curvature allows neglect of surface tension forces. Viscous momentum transfer in the flow direction is neglected.

With reference to Figure 1, the governing equations of motion thus reduce to

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} + f \tag{1}$$

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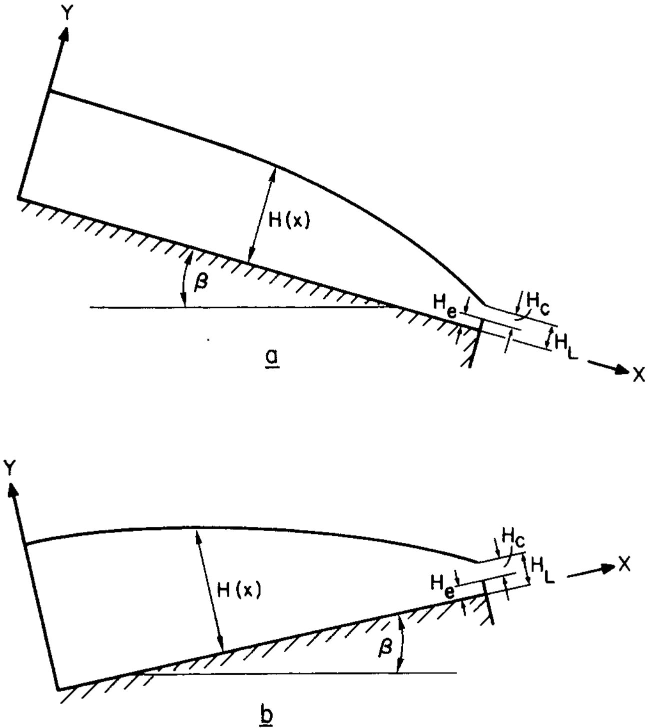

Fig. 1. Single phase flow on slightly inclined planes; *a*, positive slope; *b*, negative slope.

where

$$j = g \cdot \sin \beta \text{ for a positive slope}$$

$$j = - g \cdot \sin \beta \text{ for a negative slope}$$

$$v = - \int_0^H \frac{\partial u}{\partial x} \, dy \tag{2}$$

and

$$p = \rho \cdot g \, (H - y) \cdot \cos \beta; \quad H = H(x) \tag{3}$$

Integrating Equation (1) between the limits $0 \leq y \leq H(x)$ and introducing Equation (3), we get

$$\int_0^{H(x)} u \frac{\partial u}{\partial x} dy + \int_0^{H(x)} v \frac{\partial u}{\partial y} dy =$$

$$- g \cos \beta \, H \frac{dH}{dx} + \nu \left[\frac{\partial u}{\partial y} \right]_0^{H(x)} + j \, H \sin \beta \tag{4}$$

Now, the left-hand side of Equation (4) may be substituted by (6)

$$\int_0^{H(x)} u \frac{\partial u}{\partial x} dy + \int_0^{H(x)} v \frac{\partial u}{\partial y} dy$$

$$= \frac{\partial}{\partial x} \int_0^{H(x)} u^2 \, dy - u_f \frac{\partial}{\partial x} \int_0^{H(x)} u \, dy \tag{5}$$

where u_f denotes the interfacial velocity. [Note that the last term in Equation (5) vanished since $\int_0^{H(x)} u \, dy = \bar{u} \, H(x) = \Gamma = \text{const.}$]

Introducing

$$\left. \frac{\partial u}{\partial y} \right|_{y=H} = 0; \quad \left. \frac{\partial u}{\partial y} \right|_{y=0} = 3 \frac{\bar{u}}{H} \tag{6}$$

and

$$\int_0^H u^2 \, dy = \frac{6}{5} \, \bar{u} \, H \tag{7}$$

we get, resulting from the assumed semiparabolic velocity into Equation (4)

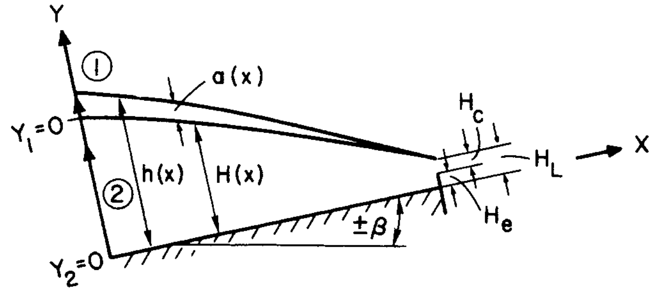

Fig. 2. Schematic presentation of coordinates for negative two-phase flow.

$$\frac{dH}{dx} = \frac{3\nu\Gamma - j \, H^3}{\frac{6}{5} \Gamma^2 - g \, H^3 \cos \beta} \tag{8}$$

The derivative dH/dx is always negative for the negatively inclined plane. For $\beta = 0$, Equation (9) reduces to Nedderman's solution for horizontal planes; that is

$$\frac{6 \, \Gamma^2 \, H}{5} - \frac{g \, H^4}{6} = 3 \, \Gamma \nu x + \text{const} \tag{9}$$

POSITIVE AND NEGATIVE SLOPES

For the inclined plane, Equation (8) yields

$$dx = dH \frac{1}{\tan \beta} \left[\frac{\zeta \mp \psi^3}{\psi^3 \mp H^3} - 1 \right] \begin{matrix} - \text{ positive } \beta & (10a) \\ + \text{ negative } \beta & (10b) \end{matrix}$$

where

$$\zeta = \frac{6}{5} \frac{\Gamma^2}{g \cos \beta}; \quad \psi^3 = \frac{3 \, \nu \, \Gamma}{g \sin \beta} \tag{11}$$

Integration of Equation (10) by parts yields a constant which can be evaluated by utilizing the boundary condition at the end of the channel where the film thickness is assumed to attain the critical height H_c (7). Here a more general form of this boundary condition

$$H_L = H_c + H_e \quad \text{at} \quad x = L \tag{12}$$

is used, where H_e is an arbitrary elevation of the channel's end, yielding

$$L - x = \frac{1}{\tan \beta} \Bigg[H - H_L + \left(\frac{\zeta \mp \psi^3}{\psi^2} \right) \left\{ \pm \frac{1}{3} \ln \left(\frac{\psi \mp H}{\psi \mp H_L} \right) \mp \frac{1}{6} \ln \left(\frac{H^2 \pm \psi \, H + \psi^2}{H_L \pm \psi \, H_L + \psi^2} \right) + \frac{1}{\sqrt{3}} \left(\arctan \left[\frac{1}{\sqrt{3}} \left(\frac{H_L}{\psi/2} - 1 \right) \right] - \arctan \left[\frac{1}{\sqrt{3}} \left(\frac{H}{\psi/2} - 1 \right) \right] \right) \right\} \Bigg] \tag{13}$$

where the upper and lower case signs relate to the positive and negative slopes, respectively.

TWO-PHASE FLOW

We consider the case of two immiscible stratified liquid films flowing concurrently on a solid plane with a slight inclination. The two liquids differ in their physical properties. Consistent with practice, the upper, lighter, and volatile films are assumed to be relatively thin, say less than one-tenth of the thickness of the major lower phase.

With reference to Figure 2, we assume the flow to be fully developed and stable, the x direction motion dominating the phenomenon. The velocity distribution in the upper and lower phase are assumed to be quasi semiparabolic and modified parabolic, respectively. Viscous momentum in the flow direction and surface tension are neglected. The governing equations reduce to

$$u_i \frac{\partial u_i}{\partial x} + v_i \frac{\partial u_i}{\partial y} = -\frac{1}{\rho_i}\frac{\partial p_i}{\partial x} + \nu_i \frac{\partial^2 u_i}{\partial y^2} + j \quad i = 1, 2 \tag{14}$$

$$p_1 = \rho_1 g (h - y) \cos\beta; \quad H \leq y \leq h \tag{15}$$

$$p_2 = \rho_2 g (H - y) \cos\beta + \rho_1 g (h - H) \cos\beta; \quad 0 \geq y \geq H \tag{16}$$

$$u_1 = u_{2f} + \bar{u}_1 - u_{2f} \frac{3}{2}\left[2\frac{y - H}{h - H} - \frac{(y - H)^2}{(h - H)^2}\right] \tag{17}$$

$$u_2 = \bar{u}_2 \left[c\frac{y}{H} + d\frac{y^2}{H^2}\right] \tag{18}$$

where the constants c and d are yet to be evaluated.

Substituting Equations (15) and (16) in (14) and integrating along the y direction, we get

$$\frac{\partial}{\partial x}\int_H^h u_1^2\, dy = -g (h - H) \cos\beta \frac{dh}{dx} + \nu_1 \left[\frac{\partial u_1}{\partial y}\right]_H^h + j (h - H) \tag{19}$$

$$\frac{\partial}{\partial x}\int_H^h u_1^2\, dy = -g H \cos\beta \frac{dH}{dx} + \frac{\rho_1}{\rho_2} g \cos\beta\, H \frac{d}{dx}(h - H) + \nu_2 \left[\frac{\partial u_2}{\partial y}\right]_0^H + jH \tag{20}$$

We now proceed to solve the constants c and d which are required for the solution of Equations (19) and (20). From the relation $\int_0^H u_2\, dy = \Gamma_2$, we obtain

$$d = 3\left(1 - \frac{c}{2}\right) \tag{21}$$

and from the interfacial shear stress through Equations (17) and (18), namely

$$\mu_2 \frac{\partial u_2}{\partial y}\bigg|_{y=H} = \mu_1 \frac{\partial u_1}{\partial y}\bigg|_{y=H} = \mu_2 \frac{\bar{u}_2}{H}(6 - 2c) \tag{22}$$

the constant c can be evaluated. The interfacial shear may be estimated through the integral momentum equation of the upper film, Equation (19). For the upper thin layers with the liquid-liquid interfacial velocity u_{2f} derived from Equation (18) when $y = H$, the left-hand side of Equation (19) becomes

$$\frac{\partial}{\partial x}\int_H^h u_1^2\, dy \simeq \frac{\partial}{\partial x}\int_H^h u_{2f}^2\, dy = \left(3 - \frac{c}{2}\right)^2 \frac{d}{dx}[\bar{u}_2^2 (h - H)] \tag{23}$$

Upon substitution, Equation (19) yields

$$\mu_1 \frac{\partial u_1}{\partial y}\bigg|_{y=H} = \rho_1 \left\{-\left(3 - \frac{c}{2}\right)^2 \frac{d}{dx}[\bar{u}_2^2 (h - H)] - g(h - H)\cos\beta \frac{dh}{dx} + j (h - H)\right\} \tag{24}$$

For small inclination angles

$$\sin\beta \simeq \beta, \quad \cos\beta \simeq 1$$

and Equation (22) becomes

$$\mu_2 \frac{\bar{u}_2}{H}(6 - 2c) = \rho_1 \left\{-\left(3 - \frac{c}{2}\right)^2 \frac{d}{dx}\left[(w - 1)\frac{\Gamma_2^2}{H}\right] - g (h - H)\left(-\frac{d(wH)}{dx} + \beta'\right)\right\} \tag{25}$$

where

$$\frac{d}{dx}[\bar{u}_2^2 (h - H)] = \frac{d}{dx}\left[(w - 1)\frac{\Gamma_2^2}{H}\right];$$

$$\frac{dh}{dx} = \frac{d}{dx}(wH) \tag{26}$$

and

$\beta' = +\beta$ for positive inclinations

$\beta' = -\beta$ for negative inclinations

For thin upper films a rather good first approximation for w is obtained by taking

$$w = 1 + \frac{\Gamma_1}{\Gamma_2}\frac{\bar{u}_1}{\bar{u}_2} \simeq 1 + \frac{\Gamma_1}{\Gamma_2}\frac{\bar{u}_{2f}}{\bar{u}_2} = 1 + \frac{2\Gamma_1}{3\Gamma_2} = \text{const} \tag{27}$$

The momentum equation of the lower phase, Equation (20), may now be solved to the first approximation by calculating the constant c from Equation (25) at any x, when w is taken from Equation (27) and dH/dx is approximated from the single-phase flow. Moreover, with laminar flow on nearly horizontal planes, the change in kinetic energy of the upper film is rather small (especially for flows on negatively inclined planes) and may be neglected. Thus, via Equation (19), Equation (25) reduces to

$$\mu_2 \frac{\partial u_2}{\partial y}\bigg|_{y=H} = \mu_2 \frac{\bar{u}_2}{H}(6 - 2c) \simeq \rho_1 w g (h - H)\left(-\frac{dH}{dx} + \beta'\right) \tag{28}$$

Note that $\left(-\frac{dH}{dx} + \beta'\right)$ designates the effective inclination angle toward the horizontal of the liquid-liquid interface at any x.

Equations (18), (21), and (28) finally yield

$$c \simeq 3 - w\frac{H \rho_1 (h - H)\, g\left(3\nu_2\Gamma_2 - \frac{6}{5}\Gamma_2^2\beta'\right)}{2\mu_2 \bar{u}_2\left(g H^3 - \frac{6}{5}\Gamma_2^2\right)} \equiv 3 - \varphi \tag{29}$$

where dH/dx is approximated from the single-phase flow, and w is given by Equation (27). Equation (18) becomes

$$\frac{u_2}{\bar{u}_2} = (3 - \varphi)\frac{y}{H} - \frac{3}{2}(1 - \varphi)\frac{y_2}{H^2} \tag{30}$$

Equation (20) can now be integrated. By utilizing Equation (5), the integrated left-hand side of Equation (20) becomes

$$\frac{\partial}{\partial x}\int_0^H u_2^2\,dy = \frac{\partial}{\partial x}[\bar{u}_2\,H\,\phi] \tag{31}$$

where

$$\phi \equiv \frac{6}{5} + \frac{1}{10}\varphi + \frac{1}{5}\varphi^2$$

Integration of the second-order derivative in Equation (20) yields

$$\left.\frac{\partial u_2}{\partial y}\right|_{y=H} - \left.\frac{\partial u_2}{\partial y}\right|_{y=0} = -3\frac{\bar{u}_2}{H}(1-\varphi) \tag{32}$$

Consistent with Equation (27), we have

$$\frac{d}{dx}(h-H) \simeq (w-1)\frac{dH}{dx} \tag{33}$$

By utilizing Equations (31), (32), and (33), Equation (20) finally reduces to

$$\frac{dH}{dx} = \frac{3\,\nu_2\,(1-\varphi)\,\Gamma_2 + jH^3}{\phi\,\Gamma_2^2 - \delta\,gH^3\cos\beta} \tag{34}$$

where

$$\delta \equiv 1 + \gamma\,(w-1)$$

Equation (34) may now be used to get a better approximation for dH/dx in Equation (28).

Equation (34) can be integrated numerically. However, since $gH^3 >> \frac{6}{5}\Gamma_2^2$ for flow on a negative slope (except at the very end of the plate), a rather good approximation can be obtained analytically. In this case, Equation (29) reduces to

$$c \simeq 3 - \gamma\,(w-1)\,w\left(\frac{3}{2} + \frac{3}{5}\frac{\Gamma_2}{\nu_2}\beta\right) \equiv 3 - \varphi' \tag{35}$$

and Equation (30) thus remains unchanged,, except that now φ' substitutes φ.

By taking $\bar{u}_1 \simeq u_{2f}$ and by utilizing Equation (30), Equation (27) yields

$$w \simeq 1 - \frac{3}{2\gamma\Omega} + \sqrt{\frac{9}{4\gamma^2\Omega^2} + \frac{2\,\Gamma_1}{\gamma\Omega\Gamma_2}} \tag{36}$$

where

$$\Omega \equiv \frac{3}{2} + \frac{3}{5}\frac{\Gamma_2}{\nu_2}\beta$$

By assuming that the upper thin layer does not affect the critical height downstream, integration of Equation (34) yields

$$\underset{\text{two phase}}{L \;-\; x} = \underset{\text{single phase}}{\delta\,[\,L - x\,]} \tag{37}$$

where now

$$\zeta \equiv \frac{\phi\,\Gamma_2^2}{\delta\,g\cos\beta};\quad \psi^3 \equiv \frac{3\,\nu_2\,(1-\varphi')\,\Gamma_2}{g\sin\beta}$$

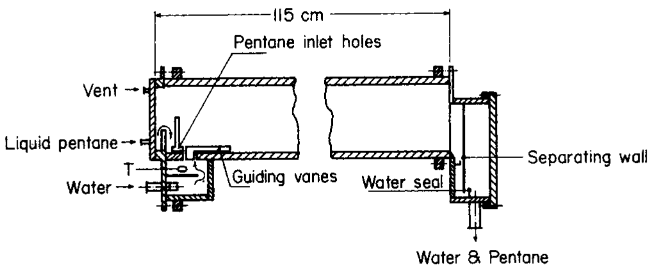

Fig. 3. Flow channel.

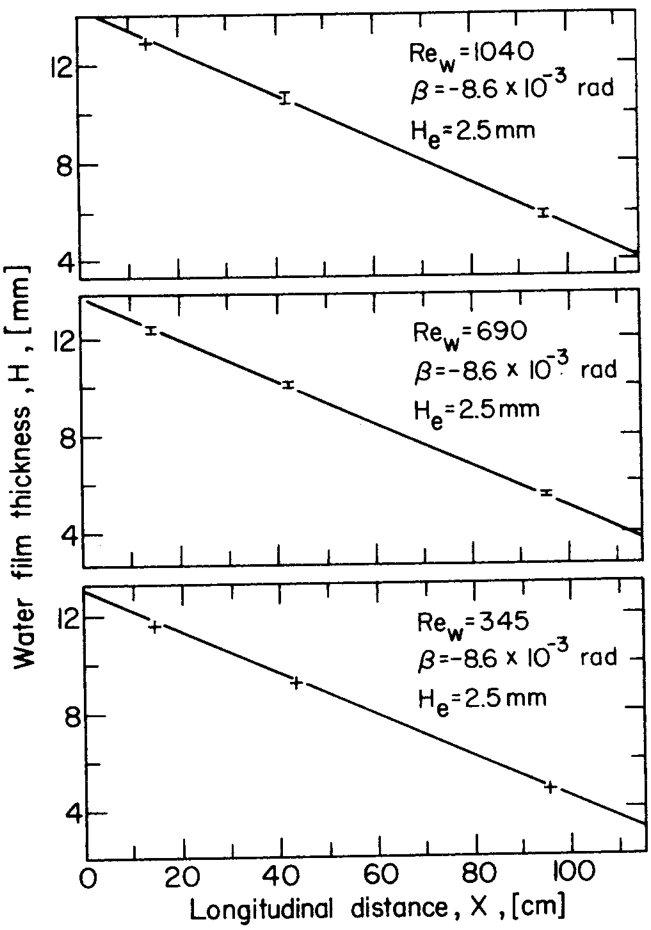

Fig. 4. Measured and calculated water film thickness on a negatively inclined plane.

EXPERIMENTAL

The flow channel, Figure 3, was a 20 cm. wide flexiglass duct, 115 cm. long. Distilled water was used for the single-phase flow runs, with *n*-pentane added for the two-phase flow study. The pentane was discharged through a row of fifteen holes, 1.5-mm. diameter. To improve pentane film stability, a row of eight longitudinal guiding vanes were inserted at the water inlet slot.

The pentane and water film thickness were measured at three points by using needle tipped 0.2-mm. diameter micrometers. A simple arrangement yielded an electric signal once the water-pentane interface was touched. The solid plane was used as a reference measurement.

Interfacial velocities were determined by following and measuring the velocities of very small bubbles.

RESULTS AND CONCLUSIONS

It is to be noted that in the whole range of operation, up to N_{Re_w} = 1,090, the flow was laminar, stable, and apparently waveless.

Comparison of measured and calculated film thickness in single-phase flow on a negative slope is presented in Figure 4. The agreement is evidently very satisfactory. As shown in Figure 5, the calculated interfacial velocities for single-phase flow fall within the range of the experimental spread of the data which consist of at least fifteen measurements.

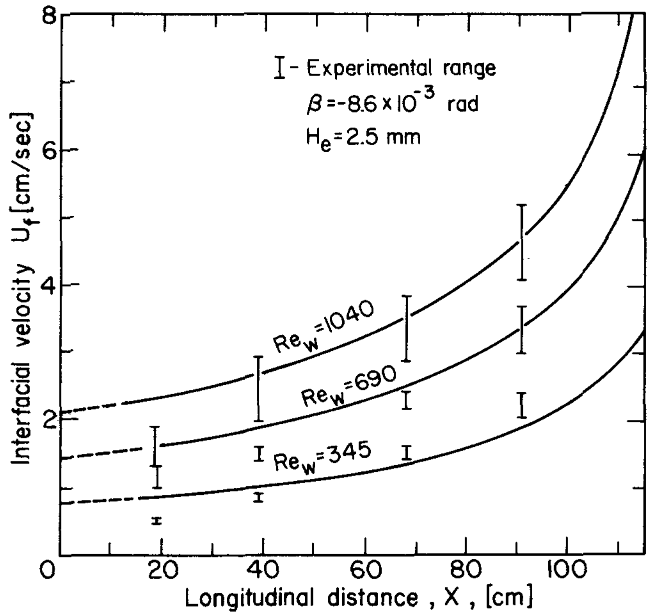

Fig. 5. Measured and calculated interfacial velocities for water film flow.

A representative comparison of the theoretical and experimental film thickness in two-phase flow on a negatively inclined plane is given in Figure 6.

The agreement of the experimental data with the theory substantiates the assumed parabolic velocity profile in film flow on a nearly horizontal plane with a negative slope. This is evidently so for single films as well as for the two-phase film flow. It is, however, important to note that the two-phase flow solution is limited to very thin upper films. Fortunately, this is the practically interesting case, and the solution presented here is presently successfully utilized in the corresponding heat transfer problem.

ACKNOWLEDGMENT

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NOTATION

c = constant, Equation (18)
d = constant, Equation (18)
g = gravitational acceleration
H = lower film thickness, $H(x)$
H_c = critical height of the major film at the end of the plate
H_e = arbitrary height of the plate at $x = L$
H_l = film total height at the end of the plate
h = total thickness of films, $(H + a)$
j = body force in the direction of flow, Equation (1)
L = length of flow section
p = pressure
N_{Re_w} = Reynolds number, water film
u = x component of velocity
$\bar{u}$ = average x component of velocity
v = y component of velocity
u_f = velocity of free interface, in single-phase flow
u_{2f} = velocity of liquid-liquid interface
x = Cartesian coordinate
y = Cartesian coordinate

Greek Letters

β, β' = inclination angle of flow to horizontal
Γ = specific flow per unit width

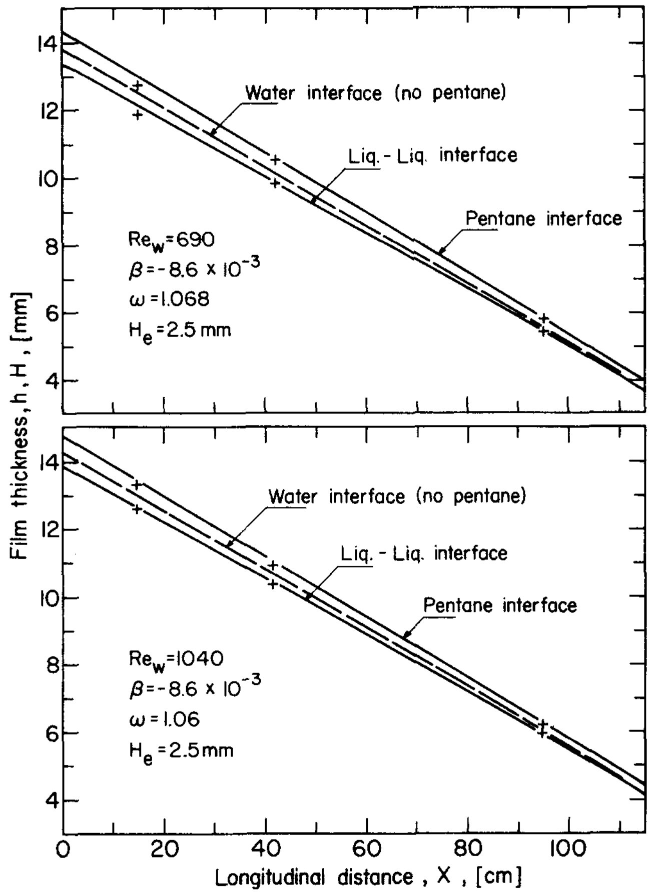

Fig. 6. Film thicknesses in two-phase flow, negatively inclined plane.

γ = ratio of densities, (ρ_1/ρ_2)
δ = constant, Equation (34)
ζ = constant, Equation (10)
μ = dynamic viscosity
ν = kinematic viscosity
ρ = density
ϕ = variable, Equation (31)
φ = constant, Equation (29)
φ' = constant, Equation (35)
ψ = constant, Equation (10)
Ω = constant, Equation (36)
ω = dimensionless height, (h/H)

Subscripts

1 = upper phase
2 = lower phase

LITERATURE CITED

1. Bentwich, M. and Samuel Sideman, *J. Heat Transfer*, **86c**, 476 (1964).
2. Gollan, A., and Samuel Sideman, *Proj. CE-37a, Rept. No. 18*, Israel Natl. Council for R & D, (Sept., 1967); *Intern. J. Heat Mass Transfer*, **11**, 1761 (1968).
3. ———, *Proj. CE-37, Rept. No. 23*, Israel Natl. Council for R & D (Apr., 1968).
4. Nedderman, R. M., *Chem. Eng. Sci.*, **21**, 715 (1966).
5. Goodridge, F., and G. Gartside, *Trans. Inst. Chem. Engrs.*, **43**, T, 62 (1965).
6. Goldstein, S., "Modern Developments in Fluid Dynamics," Oxford Univ. Press, England (1957).
7. Chow, V. T., "Open Channel Hydraulics," McGraw Hill, New York (1959).